Closed-form solution of linear buckling for tapered tubular columns with constant thickness

Marco Fabiani

Dipartimento di Ingegneria Civile, Edile e Architettura, DICEA, Università Politecnica delle Marche, Via Brecce Bianche, I-60131 Ancona, Italy

E-mail: ing.marco.fabiani@gmail.com

Keywords: linear buckling, non-uniform column, tubular, tapered, closed-form solution.

SUMMARY. In this paper an approximate analytical solution of linear buckling for tapered column with tubular cross section is presented. The thickness is constant along the column. The solution is derived in terms of Bessel functions. Numerical analysis are performed to validate the effectiveness of the solutions. It is shown that the presented approach can be used for buckling problems of any non-uniform columns provided that flexural stiffness is a strictly monotonic function. In Appendix the algorithm of numerical analysis is presented.

1 INTRODUCTION

The problem of linear buckling of columns is an old and well-studied problem, but it is still receiving attention in literature. The early works can be dated back to 18th century, when Euler discussed the problem of critical load of uniform and non-uniform columns [1]. It was only after two centuries that Gere and Carter [2] proposed a method for solving the problem of linear buckling for tapered columns with generic cross-section. The solutions presented therein were in closed-form, but no solutions were given for tapered columns with tubular cross-section. Moreover, among the recent methods that can solve the problem of linear buckling of columns with variable moment of inertia, no closed-form solutions can be found [3,4,5].

The aim of this paper is to obtain an approximate closed-form solution by a new approach for tubular column with constant thickness. Moreover, the proposed approach can be used also to evaluate approximate critical buckling load for any practical non-homogeneous columns provided that flexural stiffness is a strictly increasing function. The solution is expressed in terms of Bessel functions. An alternative numerical method is used to verify the obtained analytical results.

2 FORMULATION OF THE PROBLEM

The basic equation for the analysis of buckling of an elastic column is

$$\frac{d^2}{dx^2}\left(E(x)I(x)\frac{d^2v(x)}{dx^2}+Pv(x)\right)=0 \qquad 0\le x\le L \tag{1}$$

where $E(x)I(x)$ is the flexural stiffness along the longitudinal axes of column, $v(x)$ is the transversal displacements, P is the axial load and L is length of column. Letting $E(x)I(x) = d_0D(x)$ where d_0 is a constant value and $D(\xi)$ describes the non-uniformity of flexural rigidity of column, the following dimensionless variable $\xi = x/L$ is defined and so the Eq.(1) becomes:

$$\frac{d^2}{d\xi^2}\left(D(\xi)\frac{d^2v(\xi)}{d\xi^2}+\frac{PL^2}{d_0}v(\xi)\right)=0 \quad 0\le\xi\le 1 \tag{2}$$

The solution of Eq.(2) has the following form [6]

$$v(\xi)=c_1V_1(\xi)+c_2V_2(\xi)+c_3\xi+c_4 \tag{3}$$

where c_1, c_2, c_3, c_4 are unknowns and $c_1V_1(\xi)+c_2V_2(\xi)$ is the general solution satisfying

$$D(\xi)\frac{d^2v(\xi)}{d\xi^2}+\frac{PL^2}{d_0}v(\xi)=0 \tag{4}$$

Using the coordinate system shown in Figure 1, the moment of inertia of tapered tubular column with constant thickness is

$$I(x)=\frac{\pi}{4}[(R_0+c\,x)^4-(r_0+c\,x)^4] \tag{5}$$

Let $m=s/r_0$, $q=c\,L/r_0$, then Eq.(5) can be transformed into following expression

$$I(\xi)=\frac{\pi}{4}r_0^4\,[(1+m+q\,\xi)^4-(1+q\,\xi)^4] \tag{6}$$

so, considering that material is homogeneous, the function $D(\xi)$ is defined

$$D(\xi)=\ (1+m+q\,\xi)^4-(1+q\,\xi)^4 \tag{7}$$

and the Eq.(4) becomes

$$[(1+m+q\,\xi)^4-(1+q\,\xi)^4]\frac{d^2v(\xi)}{d\xi^2}+\alpha\,v(\xi)=0 \tag{8}$$

where α is the non-dimensional critical buckling load which is

$$\alpha=\frac{PL^2}{d_0}=\frac{4\,PL^2}{\pi E r_0^4} \tag{9}$$

Since the solution of Eq.(8) is very hard to calculate in closed-form and it is not very handy for engineering applications, an approximate method is proposed to solve Eq.(8) in closed-form.

3 SOLUTION OF THE PROBLEM

The proposed approach consists in evaluating an alternative approximate flexural stiffness that fits very well exact flexural stiffness. The form of approximate flexural rigidity is

$$D_p(\xi)=(a+b\,\xi)^k \tag{10}$$

To evaluate the three parameters a, b, k, we impose that the function in Eq.(10) is exactly satisfied at three different points of $D(\xi)$ as shown in Figure 2. The following system is obtained

$$\begin{cases} (a)^k = D(0) \\ (a+b)^k = D(1) \\ (a+b\,0.5)^k = D(0.5) \end{cases} \tag{11}$$

The Eq.(11) can be reduced into following system of equations

$$\begin{cases} a = \sqrt[k]{D(0)} \\ b = \sqrt[k]{D(1)} - \sqrt[k]{D(0)} \\ \sqrt[k]{D(0.5)} - 0.5\left(\sqrt[k]{D(1)} + \sqrt[k]{D(0)}\right) = 0 \end{cases} \tag{12}$$

The third equation can be evaluated by any root-finding algorithm such as Newton-Raphson method. For homogeneous tapered tubular column with constant thickness the following system of equation is obtained

$$\begin{cases} (a)^k = (1+m)^4 - 1 \\ (a+b)^k = (1+m+q)^4 - (1+q)^4 \\ (a+b\,0.5)^k = (1+m+q\,0.5)^4 - (1+q\,0.5)^4 \end{cases} \tag{13}$$

Considering the limitations $0 < m < 1$ and $0 < q < 5$, the numerical results of Eq.(13) gives the following expression

$$\begin{cases} k = f(m,q) > 3 & m \geq 0.05 \\ k \cong 3 & m < 0.05 \end{cases} \tag{14}$$

hence

$$a = \sqrt[k]{(1+m)^4 - 1} \tag{15}$$

and

$$b = \sqrt[k]{(1+m+q)^4 - (1+q)^4} - \sqrt[k]{(1+m)^4 - 1} \tag{16}$$

Considering the Eq.(10), (14), (15) and (16), the Eq.(8) becomes

$$\left[\sqrt[k]{(1+m)^4 - 1} + \left(\sqrt[k]{(1+m+q)^4 - (1+q)^4} - \sqrt[k]{(1+m)^4 - 1}\right)\xi\right]^k \frac{d^2 v(\xi)}{d\xi^2} + \alpha\, v(\xi) = 0 \tag{17}$$

The Eq.(17) can be exactly integrated. In fact, let

$$\bar{b}_k = \frac{b}{a} \tag{18}$$

$$t = 1 + \bar{b}_k \xi \tag{19}$$

the Eq.(17) becomes

$$t^k \frac{d^2 v(t)}{dt^2} + \beta_{p,k} v(t) = 0 \tag{20}$$

$$\beta_{p,k} = \frac{\alpha}{\bar{b}_k^2 [(1+m)^4 - 1]} \tag{21}$$

The solution of Eq.(20) is [7]

$$v(t) = \sqrt{t}\left[c_1 J_{\frac{1}{K}}\left(\frac{2}{K}\sqrt{\beta_{p,k}\, t^K}\right) + c_2 Y_{\frac{1}{K}}\left(\frac{2}{K}\sqrt{\beta_{p,k}\, t^K}\right)\right] \tag{22}$$

where $K = -k + 2$ and $J_{\frac{1}{K}}$ and $Y_{\frac{1}{K}}$ are respectively the Bessel functions of first and second type of order $1/K$. Then the complete solution is

$$v(\xi) = \sqrt{1 + \bar{b}_k \xi}\left[c_1 J_{\frac{1}{K}}\left(\frac{2}{K}\sqrt{\beta_{p,k}\left(1 + \bar{b}_k \xi\right)^K}\right) + c_2 Y_{\frac{1}{K}}\left(\frac{2}{K}\sqrt{\beta_{p,k}\left(1 + \bar{b}_k \xi\right)^K}\right)\right] + c_3 \xi + c_4 \tag{23}$$

For the sake of simplicity, the lower bound of Eq.(14) is assumed

$$k = 3 \tag{24}$$

Hence the complete solution of buckling of any tapered tubular columns becomes

$$v(\xi) = \sqrt{1 + \bar{b}_3 \xi}\left[c_1 J_1\left(2\sqrt{\frac{\beta_{p,3}}{(1 + \bar{b}_3 \xi)}}\right) + c_2 Y_1\left(2\sqrt{\frac{\beta_{p,3}}{(1 + \bar{b}_3 \xi)}}\right)\right] + c_3 \xi + c_4 \tag{25}$$

where

$$\bar{b}_3 = \frac{b}{a} = \sqrt[3]{\frac{(1+m+q)^4 - (1+q)^4}{(1+m)^4 - 1}} - 1 \tag{26}$$

and

$$\beta_{p,3} = \frac{\alpha}{\bar{b}_3^2 [(1+m)^4 - 1]} \tag{27}$$

4 RESULTS AND DISCUSSION

In this section numerical results are given for demonstrating the validity of proposed approximate moment of inertia in Eq.(10) for classical boundary conditions such as free-clamped (F-C), hinged-hinged (H-H), clamped-hinged (C-H) and clamped-clamped (C-C). The $\alpha_{ex}, \alpha_{p,k}, \alpha_{p,3}$ values are the non-dimensional critical buckling loads related to columns with exact moment of inertia (Eq.(7)), approximated moment of inertia with value of k obtained by Eq.(13), and approximated moment of inertia with value of $k = 3$, respectively. Moreover the buckling load of columns with approximate moment of inertia of thin-walled tubes is evaluated, namely α_{tw}, to show the effectiveness of the proposed method. In this case the moment of inertia is defined by following expression

$$I_{tw}(x) = \pi\, s \left(\frac{R_0 + r_0}{2} + c\, x\right)^3 \tag{28}$$

The values of α_{ex} and α_{tw} are obtained numerically by aid of *Wolfram Mathematica* ® software with an algorithm that is shown in Appendix. Then

$$\alpha_{p,k} = \beta_{p,k}\, \bar{b}_k^2 a^k \tag{29}$$

$$\alpha_{p,3} = \beta_{p,3}\, \bar{b}_3^2 a^3 \tag{30}$$

$\beta_{p,k}$ and $\beta_{p,3}$ are obtained by satisfying the appropriate boundary conditions and by imposing the determinant of the governing matrix to be zero. To verify the effectiveness of the proposed method, the following percentage errors are defined

$$\varepsilon_{p,k} = 100\, \frac{\alpha_{ex} - \alpha_{p,k}}{\alpha_{ex}} \tag{31}$$

$$\varepsilon_{p,3} = 100\, \frac{\alpha_{ex} - \alpha_{p,3}}{\alpha_{ex}} \tag{32}$$

$$\varepsilon_{tw} = 100\, \frac{\alpha_{ex} - \alpha_{tw}}{\alpha_{ex}} \tag{33}$$

The results are shown in the following tables.

s/r_0	0.05			0.1			0.5		
q	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}
0.1	0.0000	0.0000	0.0522	0.0000	-0.0002	0.1995	0.0000	-0.0033	3.4701
0.5	0.0000	-0.0011	0.0354	0.0000	-0.0039	0.1360	0.0007	-0.0509	2.4982
1.0	0.0000	-0.0024	0.0258	0.0000	-0.0089	0.0993	0.0005	-0.1248	1.8753
2.0	-0.0002	-0.0043	0.0178	-0.0010	-0.0163	0.0689	-0.0096	-0.2430	1.3136
5.0	-0.0016	-0.0070	0.0118	-0.0060	-0.0263	0.0455	-0.0777	-0.4208	0.8462

Table 1: The percentage errors $\varepsilon_{p,k}$, $\varepsilon_{p,3}$ and ε_{tw} for F-C columns.

s/r_0	0.05			0.1			0.5		
q	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}
0.1	0.0000	-0.0001	0.0543	0.0000	-0.0003	0.2069	0.0000	-0.0043	3.5748
0.5	0.0000	-0.0013	0.0411	0.0000	-0.0050	0.1574	-0.0008	-0.0662	2.8298
1.0	-0.0001	-0.0031	0.0326	-0.0006	-0.0116	0.1252	-0.0064	-0.1616	2.3015
2.0	-0.0007	-0.0055	0.0246	-0.0026	-0.0208	0.0946	-0.0317	-0.3111	1.7661
5.0	-0.0026	-0.0085	0.0169	-0.0096	-0.0323	0.0650	-0.1336	-0.5214	1.2082

Table 2: The percentage errors $\varepsilon_{p,k}$, $\varepsilon_{p,3}$ and ε_{tw} for H-H columns.

s/r_0	0.05			0.1			0.5		
q	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}
0.1	0.0000	0.0000	0.0543	0.0000	-0.0002	0.2070	0.0000	-0.0035	3.5761
0.5	0.0000	-0.0011	0.0415	0.0000	-0.0040	0.1589	0.0001	-0.0527	2.8491
1.0	0.0000	-0.0025	0.0355	-0.0002	-0.0091	0.1287	-0.0017	-0.1276	2.3492
2.0	-0.0005	-0.0043	0.0263	-0.0013	-0.0162	0.1009	-0.0147	-0.2431	1.8555
5.0	-0.0015	-0.0069	0.0196	-0.0058	-0.0247	0.0751	-0.0786	-0.4001	1.3682

Table 3: The percentage errors $\varepsilon_{p,k}$, $\varepsilon_{p,3}$ and ε_{tw} for C-H columns.

s/r_0	0.05			0.1			0.5		
q	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}	$\varepsilon_{p,k}$	$\varepsilon_{p,3}$	ε_{tw}
0.1	0.0000	0.0000	0.0543	0.0000	-0.0002	0.2071	0.0000	-0.0031	3.5767
0.5	0.0000	-0.0009	0.0417	0.0000	-0.0036	0.1596	-0.0006	-0.0469	2.8582
1.0	0.0000	-0.0022	0.0340	-0.0004	-0.0082	0.1303	-0.0045	-0.1146	2.3712
2.0	-0.0005	-0.0039	0.0270	-0.0018	-0.0148	0.1038	-0.0225	-0.2212	1.8971
5.0	-0.0018	-0.0062	0.0207	-0.0068	-0.0231	0.0794	-0.0950	-0.3726	1.4378

Table 4: The percentage errors $\varepsilon_{p,k}$, $\varepsilon_{p,3}$ and ε_{tw} for C-C columns.

The non-dimensional critical buckling loads with the proposed approach are closer to exact values than those obtained with moment of inertia of thin-walled tubes. The effectivness of proposed method is not affected by assumption of $k = 3$. For $q < 2$, that is of major interest of structural design, the percentage errors can be considered negligible. For $q > 2$, the percentage errors with proposed approach are always smaller than the percentage errors related to thin-walled tubes.

5 COLUMNS WITH VARIABLE FLEXURAL STIFFNESS

In this section, the proposed approach is employed to evaluate the non-dimensional buckling load of column with flexural stiffness which is a strictly monotonic function. The results are compared with values obtained numerically by aid of *Wolfram Mathematica* ® software and they are shown in following tables.

$D(\xi) = (1 + d\,\xi)^3(1 + b\,\xi)(1 + e\,\xi)$

$d = 0.1$ $b = 0.2$ $e = 0.3$				$d = 1.0$ $b = 2.0$ $e = 3.0$			
	α_{ex}	α_p	ε_p		α_{ex}	α_p	ε_p
F-C	4.1086	4.1087	-0.0018	F-C	49.2749	49.0823	0.3909
H-H	14.2154	14.2152	0.0011	H-H	90.6276	90.0236	0.6664
C-H	29.0831	29.0832	-0.0006	C-H	186.761	186.043	0.3847
C-C	56.8688	56.8683	0.0007	C-C	369.114	367.482	0.4421

Table 6: Non-dimensional buckling load and percentage errors ε_p for columns with rectangular cross-section whose depth and width are variable and Young modulus is a linear function (see Figure 3).

$D(\xi) = (1 + d\,\xi)^{2.6}(1 + e\,\xi)^2$

$d = 0.1$ $e = 0.2$				$d = 1.0$ $e = 2.0$			
	α_{ex}	α_p	ε_p		α_{ex}	α_p	ε_p
F-C	3.7849	3.7850	-0.0005	F-C	35.4710	35.4284	-0.1206
H-H	13.4004	13.4003	0.0002	H-H	70.4124	70.2429	0.2406
C-H	27.4158	27.4159	-0.0002	C-H	144.690	144.503	0.1296
C-C	53.6079	53.6078	0.0001	C-C	284.431	283.976	0.1599

Table 7: Non-dimensional buckling load and percentage errors ε_p for columns with I-section whose depth is variable and Young modulus is a parabolic function (see Figure 3).

6 CONCLUSIONS

In this paper approximate solution of linear buckling for tapered tubular column with constant thickness is developed in terms of Bessel functions. This aim is achieved transforming the variable moment of inertia into an approximate function such as the ordinary differential equation can be integrated exactly. Moreover the analytical solution of linear buckling for thin-walled tapered tube is considered. The non-dimensional critical buckling loads are evaluated numerically for four classical boundary conditions and they are compared with each others. The results from proposed moment of inertia are in good agreement with those computed by software program and they are more accurate than those obtained with moment of inertia of thin-walled tubes. The proposed approach can be used to evaluate approximate critical buckling load for any non-uniform columns provided that flexural stiffness is a strictly monotonic function.

APPENDIX

```
(* defining moment of inertia and its derivatives *)
i[ξ_] = (* expression of moment of inertia *);
i1[ξ_] = D[i[ξ], {ξ, 1}];
i2[ξ_] = D[i1[ξ], {ξ, 1}];
(* solution of differential equations by means of NDSolve *)
sol = NDSolve[{ i[ξ] v''''[ξ] + 2 i1[ξ]v'''[ξ] + (i2[ξ] + α[ξ])v''[ξ] == 0, (* here classical
boundary conditions *), α'[ξ] == 0, v[0.5] == 1, (* for F-C case imposing
v[0] = 1 in place of v[0.5] == 1 *)}, {v, α}, {ξ, 0,1}];
(* solution *)
Plot[Evaluate[v[ξ]/.sol], {ξ,0,1}, PlotRange → All]
{α[0]}/. sol
```

REFERENCES

[1] Elishakoff I., "A selective review of direct, semi-inverse and inverse eigenvalue problems for structures described by differential equations with variable coefficients", *Arch. Comput. Method. E.*, **7(4)**, 451-526 (2000).

[2] Gere J.M., Carter WO. "Critical buckling loads for tapered columns", *J. Struct. Div.*, **128(II)**, 736-754 (1963).

[3] Eisenberger M., "Buckling loads for variable cross-section members and variable axial forces", *Int. J. Solids Struct.*, **27(2)**, 135-143 (1991).

[4] Wei D.J., Yan S.X., Zhang Z.P., Li X.F., "Critical load for buckling of non-prismatic columns under self-weight and tip force", *Mech. Res. Commun.*, **37**, 554-558 (2010).
[5] Coskun S.B., Atay M.T., "Determination of critical buckling load for elastic columns of constant and variable cross-sections using variational iteration method", *Comput. Math. Appl.*, **58**, 2260-2266 (2009).
[6] Wang C.M., Wang C.Y., Reddy J.N., *Exact solutions for buckling of structural members*, CRC Press LLC, Boca Raton (2005).
[7] Polyanin A.D., Zaitsev V.F., *Handbook of exact solutions for ordinary differential equations*, Chapman & Hall/CRC, (1998).

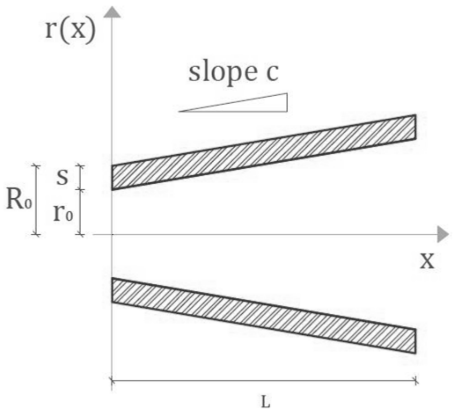

Figure 1: Coordinate system for tapered tubular columns

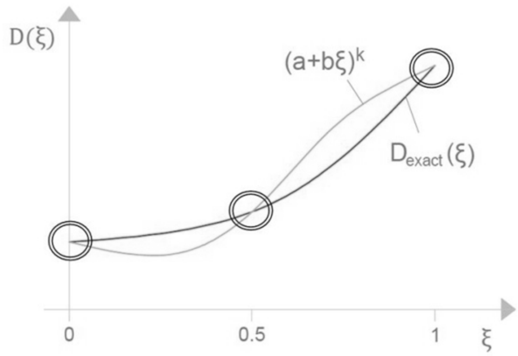

Figure 2: The exact and approximated flexural stiffness functions.

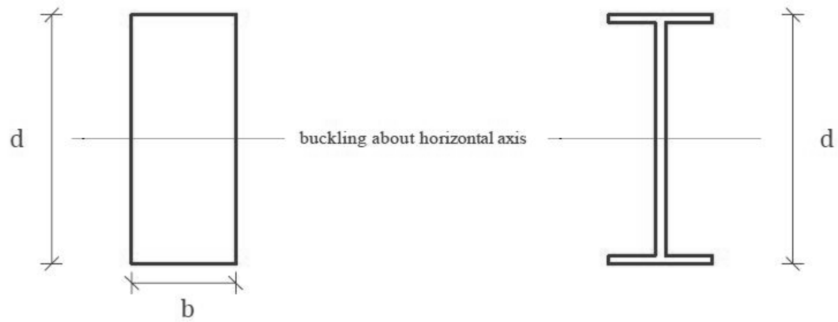

Figure 3: Cross-section of columns with variable flexural stiffness.